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Survival of a diffusing particle in an expanding cage

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Abstract
We consider a Brownian particle, with diffusion constant D, moving inside an expanding d-dimensional sphere whose surface is an absorbing boundary for the particle. The sphere has initial radius L_0 and expands at a constant rate c. We calculate the joint probability density, $p(r, t|r_0)$, that the particle survives until time t, and is at a distance r from the centre of the sphere, given that it started at a distance r_0 from the centre. The asymptotic ($t \to \infty$) probability, Q, obtained by integrating over all final positions, that the particle survives, starting from the centre of the sphere, is given by $Q = [4/\Gamma(\nu+1)\lambda^{\nu+1}]\sum_n b_n \exp\left[-\left(\alpha_\nu^n\right)^2/\lambda\right]$, where $\lambda = cL_0/D$, $b_n = \left(\alpha_\nu^n\right)^{2\nu}/\left[J_{\nu+1}\left(\alpha_\nu^n\right)\right]^2$, $\nu = (d-2)/2$ and α_ν^n is the nth positive zero of the Bessel function $J_\nu(z)$. The cases $d = 1$ and $d = 3$ are especially simple, and may be solved elegantly using backward Fokker–Planck methods.

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1. Introduction

First-passage problems for stochastic systems have attracted renewed interest in recent years [1], notably in systems with infinitely many coupled degrees of freedom, where the effective stochastic process describing a single degree of freedom is often non-Markovian [2]. However, even in systems with few degrees of freedom, and Markovian dynamics, problems with moving boundary conditions have proved difficult to solve.

In this paper we consider a single diffusing particle, or Brownian walker, moving within a d-dimensional sphere, of initial radius L_0, which is expanding at a constant rate c. The surface of the sphere defines an absorbing boundary for the particle. We present exact results, for any space dimension d, for the probability that the particle survives up to time t.

In an earlier paper [3] this problem was solved for the special case $d = 1$, in the limit $t \to \infty$, using an elegant method based on the backward Fokker–Planck equation. In section 2, we recall how this method works and show how it can be extended to the case $d = 3$. The method does not, however, appear to be useful for other values of d. Why this should be so becomes clear later in the paper, where we obtain a complete solution for general d.

To our knowledge, the results presented here (and in [3]) are the first exact results for this type of problem, although a related problem in which the cage is contracting was solved in dimension $d = 1$ using an image method [4]. However, it is not clear how this method can be extended to general dimension d. For the expanding cage, approximate methods have been developed by Krapivsky and Redner (KR) [5] in the limit of slow ('adiabatic approximation') and fast ('fast approximation') motion of the absorbing boundary. The relevant dimensionless parameter is $\lambda = cL_0/D$, where D is the diffusion constant of the particle. The slow and fast limits correspond to small and large λ respectively.

In section 3, we show that the adiabatic approximation of Krapivsky and Redner can be modified to obtain an exact solution of the usual (forward) Fokker–Planck equation for the survival probability of the particle for any time t, and any initial position within the sphere, for any space dimension d.

The outline of the paper is as follows. In section 2, we present the backward Fokker–Planck equation for the infinite-time survival probability, and its solutions for $d = 1$ and $d = 3$. In section 3 we show how the conventional forward Fokker–Planck equation can be solved to obtain the survival probability in any space dimension, for any time t and for an arbitrary starting point within the sphere. In section 4 we extract the infinite-time survival probability. The previous results for $d = 1$ and $d = 3$ are recovered as special cases. For general d, it is not straightforward to extract analytically, from the exact solution, the limiting behaviour for large λ, so in section 5 we employ the 'fast approximation' of Krapivsky and Redner to obtain an approximate solution that becomes exact in this limit. Section 6 concludes with a brief summary of the results.

2. Infinite-time survival probability of a particle in a expanding cage: the backward Fokker–Planck method

The backward Fokker–Planck equation is a powerful method for the study of first-passage problems in many contexts [6]. In this section, we show how it can be used to obtain, rather easily, the infinite-time survival probabilities in one and three dimensions. For the general case in which we wish to determine the survival probability at general time t, we need to use the usual forward Fokker–Planck equation. Since this approach is more algebraically complex, we defer it to section 3.

2.1. Solution in one dimension

In [3] we considered a diffusing particle obeying the Langevin equation $\dot{X}(t) = \eta(t)$, $X(0) = x$, where $\eta(t)$ is Gaussian white noise with mean zero and time correlator $\langle\eta(t_1)\eta(t_2)\rangle = 2D\delta(t_1 - t_2)$, bounded by a linearly-expanding absorbing cage with edges located at $\pm(L_0 + ct)$. After a time Δt the boundaries have moved to positions $\pm(L_0 + c(t + \Delta t))$ and the particle has moved to $x + \Delta x$, where $\langle\Delta x\rangle = 0$ and $\langle(\Delta x)^2\rangle = 2D\Delta t$. The probability $Q(x, L_0, t)$ that the particle still survives after a time t satisfies the obvious equation $Q(x, L_0, t) = \langle Q(x + \Delta x, L_0 + c\Delta t, t - \Delta t)\rangle$. Expanding to first order in Δt yields the backward Fokker–Planck equation

$$\frac{\partial Q}{\partial t} = D\frac{\partial^2 Q}{\partial x^2} + c\frac{\partial Q}{\partial L_0}. \tag{1}$$

The novelty in this approach resides in treating L_0, which gives the initial positions of the boundaries, as an additional independent variable.

Discarding the time-derivative term, to directly obtain the infinite-time limit, and introducing dimensionless variables $y = cx/D$ and $\lambda = cL_0/D$ gives the simplified equation

$$\frac{\partial^2 Q}{\partial y^2} + \frac{\partial Q}{\partial \lambda} = 0, \tag{2}$$

subject to the absorbing boundary conditions $Q(\pm\lambda, \lambda) = 0$, and with $Q(y, \lambda \to \infty) = 1$. The solution is

$$Q(y, \lambda) = \sum_{n=-\infty}^{\infty} (-1)^n \cosh(ny)\, \mathrm{e}^{-n^2\lambda}, \tag{3}$$

which satisfies both the differential equation and the boundary conditions.

For a particle starting at the origin ($y = 0$), the survival probability $Q(0, \lambda)$ is given by

$$Q(0, \lambda) = \sum_{n=-\infty}^{\infty} (-1)^n\, \mathrm{e}^{-n^2\lambda} \sim 1 - 2\mathrm{e}^{-\lambda}, \qquad \lambda \to \infty. \tag{4}$$

We can extract the small-λ behaviour by using the Poisson sum formula,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \tilde{f}(2\pi k),$$

where $\tilde{f}(k)$ is the Fourier transform of the function $f(n)$ to recast equation (3) in the form

$$Q(0, \lambda) = \sqrt{\frac{\pi}{\lambda}} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi^2(2k-1)^2/4\lambda} \sim 2\sqrt{\frac{\pi}{\lambda}}\, \mathrm{e}^{-\frac{\pi^2}{4\lambda}}, \qquad \lambda \to 0. \tag{5}$$

2.2. Solution in three dimensions

We now extend this result to the case of a diffusing particle in three spatial dimensions bounded by a linearly-expanding, absorbing sphere of radius $L(t) = L_0 + ct$ using the same backward Fokker–Planck method.

For general spatial dimensionality d, equation (2) has the obvious generalization

$$\nabla^2 Q + \frac{\partial Q}{\partial \lambda} = 0 \tag{6}$$

where the dimensionless spatial coordinate is $\mathbf{r} = c\mathbf{r}_0/D$, and $\lambda = cL_0/D$ as before, where $\mathbf{r}_0$ is the initial location of the particle within the sphere. Exploiting the spatial isotropy, we infer that Q depends on $\mathbf{r}$ only through its magnitude, $r = |\mathbf{r}|$, giving, for $d = 3$,

$$\frac{\partial^2 Q}{\partial r^2} + \frac{2}{r}\frac{\partial Q}{\partial r} + \frac{\partial Q}{\partial \lambda} = 0. \tag{7}$$

This equation has separable solutions of the form

$$Q_k(r, \lambda) = \frac{\sinh(kr)}{r}\, \mathrm{e}^{-k^2\lambda},$$

characterized by an index k, from which a general solution can be constructed by superposition. Note that solutions of the form $[\cosh(kr)/r]\exp(-k^2\lambda)$ are rejected as they are not regular at the origin. At this point we make no assumptions about the values of k. We find, however, that no simple superposition of solutions $Q_k(r, \lambda)$ satisfies the required boundary condition $Q(\lambda, \lambda) = 0$. We note, however, that any derivative of $Q_k(r, \lambda)$ with respect to k is also a solution of equation (7) since it is simply a superposition of two values of k which are infinitesimally close together. We therefore try taking the first derivative of this solution and postulate a sum of the functions $\mathrm{d}Q_k(r,\lambda)/\mathrm{d}k$ with integer values of k, in analogy with the one-dimensional case, to obtain

$$Q(r,\lambda) = \sum_{n=-\infty}^{\infty} a_n \left[\cosh(nr) - \frac{2n\lambda}{r}\sinh(nr)\right] \mathrm{e}^{-n^2\lambda}.$$

If we choose the amplitudes to be $a_n = 1$ for all n, we see by inspection that the boundary conditions $Q(\lambda,\lambda) = 0$ and $Q(r, \lambda \to \infty) = 1$ are satisfied. The survival probability is therefore given by

$$Q(r,\lambda) = \frac{1}{r}\sum_{n=-\infty}^{\infty} [r\cosh(nr) - 2n\lambda\sinh(nr)]\,\mathrm{e}^{-n^2\lambda}. \tag{8}$$

For a particle starting at the origin we take the limit $r \to 0$, giving

$$Q(0,\lambda) = \sum_{n=-\infty}^{\infty} (1 - 2\lambda n^2)\,\mathrm{e}^{-n^2\lambda} \tag{9}$$

$$\sim 1 - 2(2\lambda - 1)\,\mathrm{e}^{-\lambda}, \qquad \lambda \to \infty. \tag{10}$$

To obtain an expression suitable for extracting the behaviour at small λ, we take the Poisson transform of this sum, giving

$$Q(0,\lambda) = \frac{2\pi^{5/2}}{\lambda^{3/2}} \sum_{k=-\infty}^{\infty} k^2\,\mathrm{e}^{-\frac{\pi^2 k^2}{\lambda}} \sim \frac{2\pi^{5/2}}{\lambda^{3/2}}\,\mathrm{e}^{-\frac{\pi^2}{\lambda}}, \tag{11}$$

for $\lambda \to 0$.

For general space dimension d, the backward Fokker–Planck method does not seem to be useful. The reason for this will become clear in the following section.

3. Survival probability in a d-dimensional expanding sphere

We approach the problem in general dimension by solving the forward Fokker–Planck equation for the probability density, $p(\mathbf{r}, t|\mathbf{r}_0, 0)$, defined as the probability density that the particle, starting at position $\mathbf{r}_0$ within the sphere of radius L_0, still survives (has not yet reached the absorbing boundary) at time t, and is currently at position $\mathbf{r}$. It satisfies the partial differential equation $\partial p/\partial t = D\nabla^2 p$. The boundary condition is $p(\mathbf{r}, t|\mathbf{r}_0, 0) = 0$ for $|\mathbf{r}| = L_0 + ct$ and the initial condition is $p(\mathbf{r}, 0|\mathbf{r}_0, 0) = \delta^d(\mathbf{r} - \mathbf{r}_0)$.

In generalized polar coordinates, the Fokker–Planck equation reads

$$\frac{\partial p}{\partial t} = D\left(\frac{\partial^2 p}{\partial r^2} + \frac{d-1}{r}\frac{\partial p}{\partial r} + \mathcal{L}_\Omega p\right), \tag{12}$$

where $\mathcal{L}_\Omega$ is a generalized angular derivative operator. From the spherical symmetry of the problem, $p(\mathbf{r}, t|\mathbf{r}_0, 0)$ only depends on $r \equiv |\mathbf{r}|$, $r_0 \equiv |\mathbf{r}_0|$ and the angle between $\mathbf{r}$ and $\mathbf{r}_0$. The problem can be simplified by choosing the direction of $\mathbf{r}_0$ as the principal polar axis and integrating out the angular degrees of freedom by defining

$$\bar{p}(r,t) = \frac{1}{S_d}\int_\Omega \mathrm{d}\Omega\, p(\mathbf{r}, t|\mathbf{r}_0, 0),$$

where $\mathrm{d}\Omega$ is an element of solid angle, $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the integral over the solid angle, i.e. S_d is the surface area of the unit sphere in d dimensions, and the dependence of $\bar{p}(r,t)$ on r_0 is implicit.

The differential equation for $\bar{p}$ reads

$$\frac{\partial \bar{p}}{\partial t} = D\left(\frac{\partial^2 \bar{p}}{\partial r^2} + \frac{d-1}{r}\frac{\partial \bar{p}}{\partial r}\right). \tag{13}$$

The initial condition is

$$\bar{p}(r, 0) = \frac{\delta(r - r_0)}{S_d r_0^{d-1}}. \tag{14}$$

Our method of solution for the moving boundary problem is motivated by the solution for a fixed absorbing boundary at $r = L_0$. Separable solutions of equation (13), regular at the origin, have the form

$$\bar{p}_k(r, t) = \frac{J_\nu(kr)}{r^\nu}\,\mathrm{e}^{-k^2 Dt},$$

where $J_\nu(x)$ is the Bessel function of order ν and $\nu = (d-2)/2$. The absorbing boundary condition, $\bar{p}(L_0, t) = 0$, selects a discrete set of k-values, $k_n = \alpha_\nu^n \big/ L_0$, where α_ν^n is the nth positive zero of $J_\nu(x)$, to give the discrete set of solutions

$$\bar{p}_n(r, t) = \frac{J_\nu\left(\alpha_\nu^n r \big/ L_0\right)}{r^\nu}\,\mathrm{e}^{-(\alpha_\nu^n)^2 Dt/L_0^2}. \tag{15}$$

Following the approach of KR [5], our trial solution of equation (13) replaces, in equation (15), the initial radius L_0 by the time-dependent radius $L(t) = L_0 + ct$, and $t \big/ L_0^2$ by $\int_0^t \mathrm{d}t'/L^2(t')$. We also multiply the static solution by an unknown function of r and t to give

$$\bar{p}(r, t) = g(r, t)\frac{J_\nu\left(\frac{\alpha_\nu^n r}{L(t)}\right)}{r^\nu}\exp\left\{-\left(\alpha_\nu^n\right)^2 D\int_0^t \frac{\mathrm{d}t'}{L^2(t')}\right\}. \tag{16}$$

This differs from the KR approach in two ways. First, we allow a general function $g(r, t)$, whereas KR only consider functions $g(t)$, independent of r. Secondly, we will ultimately superpose all possible solutions of the form (16), with different values of n, to obtain the exact solution, whereas KR keep only the lowest mode ($n = 1$) and are therefore restricted to the adiabatic limit. On the other hand, the KR method works in the adiabatic limit for any form of cage expansion, whereas our approach is restricted, as we shall see, to a linearly expanding cage.

The alert reader will have noted that by introducing a general function $g(r, t)$ in equation (16) we have simply replaced one unknown function, $\bar{p}(r, t)$, by another, $g(r, t)$. This is, of course, true. However, the resulting equation for $g(r, t)$ simplifies greatly for the case of interest, namely $L(t) = L_0 + ct$, when it can be exactly solved.

On substituting equation (16) into equation (13) we obtain the following equation for $g(r, t)$:

$$\frac{\partial_{rr} g}{g} + \frac{1}{r}\frac{\partial_r g}{g} - \frac{1}{D}\frac{\dot{g}}{g} + \left(\frac{\alpha_\nu^n}{L}\right)\left(\frac{r\dot{L}}{DL} + 2\frac{\partial_r g}{g}\right)\frac{J_\nu'\left(\alpha_\nu^n r \big/ L\right)}{J_\nu\left(\alpha_\nu^n r \big/ L\right)} = 0, \tag{17}$$

where dots indicate time derivatives and $J_\nu'(x) = \mathrm{d}J_\nu/\mathrm{d}x$.

In general, this partial differential equation is intractable, so we try to simplify it by looking for a solution in which the terms that involve Bessel functions, and the terms that do not, separately vanish. Such a solution exists if the two equations

$$\dot{g} = D\left(\partial_{rr} g + \frac{1}{r}\partial_r g\right) \tag{18}$$

$$\partial_r g = -\left(\frac{\dot{L}r}{2DL}\right) g \tag{19}$$

are both satisfied by the same function $g(r,t)$. Equation (19) can be integrated immediately to give

$$g(r,t) = A(t)\exp\left(-\frac{\dot{L}r^2}{4DL}\right), \tag{20}$$

where $A(t)$ is an arbitrary function. Substituting this result into equation (18) gives

$$\dot{A} = \left(\frac{\ddot{L}r^2}{4DL} - \frac{\dot{L}}{L}\right)A. \tag{21}$$

By assumption, $A(t)$ depends only on t, and not on r. For a consistent solution, therefore, we require $\ddot{L} = 0$, i.e. the absorbing boundary must move at constant speed. Solving equation (21) for this case gives

$$A(t) = K/L(t), \tag{22}$$

where K is an arbitrary constant.

The general solution for the case $L(t) = L_0 + ct$ is obtained as an arbitrary superposition of separable solutions, combining equations (16), (20) and (22):

$$\bar{p}(r,t) = \sum_n \frac{a_n}{L(t)r^\nu} J_\nu\left(\frac{\alpha_\nu^n r}{L(t)}\right) \mathrm{e}^{-\frac{(\alpha_\nu^n)^2 Dt}{L_0 L(t)} - \frac{cr^2}{4DL(t)}}, \tag{23}$$

the summation being over all positive zeros of $J_\nu(x)$.

As usual, the amplitudes a_n are determined by the initial condition, equation (14), exploiting the orthogonality property of Bessel functions:

$$\int_0^{L_0} r\,\mathrm{d}r\, J_\nu\left(\frac{\alpha_\nu^n r}{L_0}\right) J_\nu\left(\frac{\alpha_\nu^m r}{L_0}\right) = \frac{L_0^2}{2}\left[J_{\nu+1}\left(\alpha_\nu^m\right)\right]^2 \delta_{nm}.$$

The exact solution is thus

$$\bar{p}(r,t) = \frac{2}{L_0 L(t)(rr_0)^\nu S_d} \sum_n \frac{J_\nu\left(\frac{\alpha_\nu^n r}{L(t)}\right) J_\nu\left(\frac{\alpha_\nu^n r_0}{L_0}\right)}{\left[J_{\nu+1}\left(\alpha_\nu^n\right)\right]^2} \mathrm{e}^{-\frac{(\alpha_\nu^n)^2 Dt}{L_0 L(t)} - \frac{c}{4D}\left(\frac{r^2}{L(t)} - \frac{r_0^2}{L_0}\right)}. \tag{24}$$

Equation (24) represents our most general result, giving the survival probability for general time t, for general values of the initial and final distances from the origin, and for general space dimensionality d. Note that $\bar{p}(r,t)$ depends on the constants L_0, c and D as well as r_0, r and t. To make contact with our earlier results for $d = 1$ and $d = 3$, we now compute the infinite-time survival probability, for given starting radius r_0, by letting $t \to \infty$ and integrating over the final coordinate r.

4. The infinite-time survival probability

To find the probability that the particle survives for infinite time, we integrate the probability density over the whole domain. We introduce the dimensionless variables, $\rho = cr_0/D$ and $\lambda = cL_0/D$, in terms of which the infinite-time survival probability can be expressed as

$$Q(\rho,\lambda) = \lim_{t\to\infty} \frac{2\mathrm{e}^{\rho^2/4\lambda}}{L_0 L(t) r_0^\nu} \sum_{n=1}^{\infty} \frac{J_\nu\left(\frac{\alpha_\nu^n \rho}{\lambda}\right)}{J_{\nu+1}^2\left(\alpha_\nu^n\right)} \mathrm{e}^{-\frac{(\alpha_\nu^n)^2}{\lambda}} I_n(t), \tag{25}$$

where

$$I_n(t) = \int_0^{L(t)} \frac{r^{d-1}\,\mathrm{d}r}{r^\nu} J_\nu\left(\frac{\alpha_\nu^n r}{L(t)}\right) \mathrm{e}^{\frac{-cr^2}{4DL(t)}}. \tag{26}$$

To extract the large-t behaviour, we change variables to $x = r/L(t)$ in the integral

$$I_n(t) = L(t)^{d-\nu} \int_0^1 x^{d-1-\nu} J_\nu\left(\alpha_\nu^n x\right) \mathrm{e}^{-\frac{cx^2 L(t)}{4D}} \, \mathrm{d}x. \tag{27}$$

For large t, i.e. large $L(t)$, the integral over x is dominated by small values of x and the Bessel function can be replaced by its small-argument form $J_\nu(z) \sim (z/2)^\nu/\Gamma(\nu+1)$, to give, for $t \to \infty$,

$$I_n(t) \sim 2^{d-1} \left(\frac{D}{c}\right)^{d/2} \left(\frac{\alpha_\nu^n}{2}\right)^\nu L(t). \tag{28}$$

Putting this result into equation (25) gives our final result

$$Q(\rho, \lambda) = \frac{2^{\nu+2}}{\rho^\nu \lambda} \mathrm{e}^{\rho^2/4\lambda} \sum_{n=1}^{\infty} \frac{\left(\alpha_\nu^n\right)^\nu J_\nu\left(\alpha_\nu^n \frac{\rho}{\lambda}\right)}{\left[J_{\nu+1}\left(\alpha_\nu^n\right)\right]^2} \mathrm{e}^{-\frac{(\alpha_\nu^n)^2}{\lambda}}, \tag{29}$$

where we recall that $\rho = cr_0/D$, $\lambda = cL_0/D$ and $\nu = d/2 - 1$.

It is interesting to consider the special case of a particle starting at the origin. Taking the limit $\rho \to 0$ in equation (29) gives

$$Q(0, \lambda) = \frac{4}{\Gamma(\nu+1)\lambda^{\nu+1}} \sum_{n=1}^{\infty} \frac{\left(\alpha_\nu^n\right)^{2\nu}}{\left[J_{\nu+1}\left(\alpha_\nu^n\right)\right]^2} \mathrm{e}^{-\frac{(\alpha_\nu^n)^2}{\lambda}}. \tag{30}$$

The limiting form for small λ is given by the first term in the sum.

We now compare our results with those obtained using backward Fokker–Planck methods in section 2. The cases $d = 1$ and $d = 3$ correspond to $\nu = -1/2$ and $\nu = 1/2$ respectively. The functions $J_{-1/2}(z)$ and $J_{1/2}(z)$ have positive zeros at $z = (2n-1)\pi/2$ and $z = n\pi$ respectively, while $J_{1/2}^2[(2n-1)\pi/2] = 4/[\pi^2(2n-1)]$ and $J_{3/2}^2[n\pi] = 2/(\pi^2 n)$. It is the fact that the Bessel functions zeros are uniformly spaced in $d = 1$ and $d = 3$ that makes these cases especially simple.

For $d = 1$, the final result, equation (29), can be written as

$$Q(\rho, \lambda) = \mathrm{e}^{\rho^2/4\lambda} \sqrt{\frac{\pi}{\lambda}} \sum_{n=-\infty}^{\infty} \cos\left((2n-1)\pi\rho/2\lambda\right) \mathrm{e}^{-(2n-1)^2\pi^2/4\lambda}, \tag{31}$$

while for $d = 3$ it takes the form

$$Q(\rho, \lambda) = \mathrm{e}^{\rho^2/4\lambda} \frac{2\pi^{3/2}}{\rho\sqrt{\lambda}} \sum_{n=-\infty}^{\infty} n \sin(n\pi\rho/\lambda) \mathrm{e}^{-n^2\pi^2/\lambda}. \tag{32}$$

For the special case $\rho = 0$, these results reduce to equations (5) and (11). One can show this correspondence holds for general ρ by using the Poisson summation formula (with $\rho \to y$ in $d = 1$ and $\rho \to r$ in $d = 3$) to transform equation (29) into equations (3) and (8) for $d = 1$ and $d = 3$ respectively.

It is not simple to use the Poisson summation formula on equation (29) for general d, since we do not have explicit expressions for α_ν^n except for $\nu = \pm 1/2$. This means that it is not straightforward to analytically extract the large-λ limit of $Q(0, \lambda)$. In the final part of this paper, therefore, we extend the 'fast approximation' of [5] to general dimension d in order to investigate the large-λ behaviour.

5. Approximate solution for fast expansion (large λ)

For a rapidly expanding cage in d dimensions we follow the approximate method used in [5]. This approach consists of equating the loss of survival probability to the outward probability flux at the boundary. For a particle starting at the origin, we define $Q(t)$ to be its survival probability at time t. If the cage is expanding rapidly, the probability flux through the boundary is small and we can use the free form for the diffusion propagator:

$$p(r,t) = Q(t)\frac{\exp(-r^2/4Dt)}{(4\pi Dt)^{d/2}},$$

where r is the distance from the origin at time t. We equate the rate of decrease of Q to the flux through the boundary:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = S_d L(t)^{d-1} D\, \partial_r p(r,t)|_{r=L(t)}$$

or

$$-\ln Q(t) = \frac{1}{\Gamma(d/2)(4D)^{d/2}} \int_0^t \mathrm{d}u \frac{(L_0+cu)^d}{u^{\frac{d}{2}+1}} \exp\left(-\frac{(L_0+cu)^2}{4Du}\right). \tag{33}$$

Setting $t = \infty$, we can evaluate the integral for large λ using the method of steepest descents to obtain the infinite-time result

$$Q(\infty) = \exp\left(-\frac{2\sqrt{\pi}\lambda^{\frac{d-1}{2}}}{\Gamma(d/2)}\,\mathrm{e}^{-\lambda}\right)$$
$$\sim 1 - \frac{2\sqrt{\pi}\lambda^{\frac{d-1}{2}}}{\Gamma(d/2)}\,\mathrm{e}^{-\lambda}, \qquad \lambda \to \infty, \tag{34}$$

which, for the cases $d = 1$ and $d = 3$, agrees with the large-λ limits given in equations (4) and (10) respectively.

6. Summary

In this paper, we have derived exact results, in arbitrary space dimension d, for the survival probability of a particle diffusing inside a uniformly expanding cage that acts as an absorbing boundary for the particle. We found very simple forms for the survival probability in one and three dimensions by using a backward Fokker–Planck approach. Our solution for general d, however, using forward Fokker–Planck methods, indicates that the simplicity of the solutions in one and three dimensions is fortuitous. Our method of solution seems to be restricted to the case where the boundary moves at constant speed, though it will be interesting to pursue the question of whether there are any other soluble cases.

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